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Abstract: Asymptotics consist in formal series of the solution to a problem which involves a small parameter. When truncated at a certain order, the finite serie provides an approximation of the exact solution with a given accuracy, and the coefficients of this sum are solution to elementary problem that do not depend on the small parameter, which can be for instance the thickness of the domain or a small or high conductivity coefficient. This a useful tool to obtain approximate expressions of the solution to the so-called Eddy Current problem, which describes the magnetic potential in a material composed by a dielectric material surrounding a conductor. However such expansions are derivatives consuming, in the sense that to go further in the expansion, it is necessary to compute the higher derivatives of the first orders terms, and it also requires a precise knowledge of the geometry, since derivatives of the parameterization of the interface dielectric/conductor are involved. From the numerical point of view, this leads to instability which may restrict or prevent a direct use of the asymptotic expansion. The aim of this report is to present a numerical way to tackle such drawbacks by using the property that the coefficients of the expansion are real of the source term is real, making it possible to identify numerically the first two terms of the expansion.

Key-words: Asymptotics expansion, Eddy Current, Finite Element Methods

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Utilisation pratique de développements asymptotiques pour résoudre numériquement le problème des courants de Foucault

Résumé : Les développements asymptotiques fournissent un outil efficace pour approcher les solutions de problèmes impliquant un petit paramètre. En particulier, dans le problème des courants de Foucault, les solutions proposées par Leontovitch, puis Senior et Volakis et étendues par Haddar, Joly et Nguyen permettent d'approcher efficacement le potential magnétique à n'importe quel ordre, du moins en théorie. Cependant le procédé du développement est couteux en dérivées : pour obtenir les termes suivants du développement, il faut calculer précisément les dérivées d'ordre supérieur des coefficients obtenus pour les approximations inférieures, ainsi que les dérivées de la paramétrisation de l'interface diélectrique/conducteur. L'application numérique de tels développements est donc souvent limitée à une approximation à l'ordre 1 car les calculs de courbure sont souvent délicats. Dans ce rapport, nous présentons une stratégie numérique qui permet d'obtenir les 3 premiers coefficients du potentiel magnétique, sans calcul de la courbure du bord du domaine. L'idée est de calculer la solution du problème complet des courants de Foucault à une fréquence "raisonnable" telle que l'épaisseur de peau δ_0 ne soit pas trop petite pour le maillage, mais telle qu'une approximation à l'ordre δ_0^3 soit assez précise. Ensuite, un calcul de la solution "limite" conducteur parfait permet de retrouver numériquement les coefficients d'ordre 1 et 2 dans le diélectrique. Ceci utilise le caractère réel des coefficients du développement lorsque la source est réelle. Nous montrons numériquement l'efficacité de la méthode et nous prouvons son fondement théorique.

Mots-clés : Développements asymptotiques, Courants de Foucault, Méthode Eléments Finis

Contents

1 Introduction

Let Ω be a smooth bounded domain of $\mathbb{R}^d$, for $d = 2, 3$. Assume that Ω is split into two domains:

$$\Omega = \Omega_- \cup \Omega_+ \cup \Gamma\,,$$

where $\Omega_- \subset \mathbb{R}^d$ is the domain of the conductor and Ω_+ being the surrounding dielectric material. We assume that the boundary of Ω_-, denoted by Γ, is smooth.

Let $\delta > 0$ be a small parameter, and let $\alpha \in \mathbb{C}$ be such that α^2 the following boundary value problem satisfied by the magnetic potential $(\mathcal{A}_\delta^+, \mathcal{A}_\delta^-)$ admits an unique solution:

$$\left\{\begin{array}{lcll} -\Delta \mathcal{A}_\delta^+ & = & J^+ & \text{in} \quad \Omega_+ \ , \\ -\Delta \mathcal{A}_\delta^- + \dfrac{\alpha^2}{\delta^2}\mathcal{A}_\delta^- & = & 0 & \text{in} \quad \Omega_- \ , \\ \mathcal{A}_\delta^+ & = & \mathcal{A}_\delta^- & \text{on} \quad \Gamma \ , \\ \partial_{\mathbf{n}} \mathcal{A}_\delta^+ & = & \partial_{\mathbf{n}} \mathcal{A}_\delta^- & \text{on} \quad \Gamma \ , \\ \mathcal{A}_\delta^+ = 0 & & & \text{on} \quad \partial\Omega \ . \end{array}\right. \tag{1}$$

Of course, any other boundary conditions on $\partial\Omega$, which ensure the well-posedness of the above problem can be imposed. It is well-known that $(\mathcal{A}_\delta^+, \mathcal{A}_\delta^-)$ have the following asymptotic expansion for $\delta \to 0$:

$$\begin{aligned} &\mathcal{A}_\delta^+ \sim \sum_{j\geqslant 0} \delta^j\ \mathcal{B}_j^+, \\ &\mathcal{A}_\delta^- \sim \sum_{j\geqslant 0} \delta^j\ \mathcal{B}_j^-(\mathbf{x};\delta) \quad \text{with} \quad \mathcal{B}_j^-(\mathbf{x};\delta) = \chi(\eta)\mathrm{w}_j(\mathbf{x_T}, \frac{\eta}{\delta}), \end{aligned}$$

where a change of variables $\mathbf{x} \to (\mathbf{x_T}, \eta)$ is performed in the conducting material in order to describe the boundary layer in which the electric field is confined, $\mathbf{x_T}$ being the tangential variable to the interface Γ and η being the normal variable in a neighborhood $\mathcal{V}(\Gamma)$ of Γ in the conductor.

The first terms of the expansion have been obtained by Leontovitch and Rytov in the 40's [3], while extensions have been obtained in the late 80's by Senior and Volakis [4]. The reader may refer to Haddar *et al.* for a mathematical justification of the expansion [1]. More precisely, Haddar *et al.* have shown that if the source J^+ is regular enough, there exists δ_0 such that for any $k \geq 0$, there exists a constant $C_k(\Omega)$

$$\left\| \mathcal{A}_\delta^+ - \sum_{l=0}^{k} \delta^l \mathcal{B}_l^+ \right\|_{H^1(\Omega^+)} \leq C_k(\Omega)\delta^{k+1}, \qquad \forall \delta \in (0, \delta_0). \tag{2}$$

In the above expansion, all the coefficients $(\mathcal{B}_j^+, \mathcal{B}_j^-)$ of the expansion are complex-valued functions, which satisfy "elementary" problems that do not depend on δ. Such an expansion is interesting since for instance the computation of the first $2k$ coefficients $(\mathcal{B}_j^+, \mathcal{B}_j^-)_{j=0,\cdots,k-1}$ - which do not involve any small parameter - makes it possible to obtain an approximation of $\mathcal{A}_\delta^+$ with an accuracy of order $O(\delta^k)$ for any δ small enough. Another advantage of such expansion holds in the fact that if δ is modified, the terms $(\mathcal{B}_j^+, \mathcal{B}_j^-)$ do not have to be recomputed. It is also well-known that $\mathcal{B}_0^+$ and $\mathcal{B}_1^+$ are given by

$$\left\{\begin{array}{lcll} -\Delta\mathcal{B}_0^+ &=& J^+ & \text{in } \Omega_+ , \\ \mathcal{B}_0^+ &=& 0 & \text{on } \Gamma, \\ \mathcal{B}_0^+ &=& 0 & \text{on } \partial\Omega , \end{array}\right. \qquad \left\{\begin{array}{lcll} -\Delta\mathcal{B}_1^+ &=& 0 & \text{in } \Omega_+ , \\ \mathcal{B}_1^+ &=& -\dfrac{1}{\alpha}\partial_{\mathbf{n}}\mathcal{B}_0^+ & \text{on } \Gamma, \\ \mathcal{B}_1^+ &=& 0 & \text{on } \partial\Omega . \end{array}\right. \tag{3}$$

However, it is difficult to use straightforwardly the expansion at higher order. Indeed, the process is "derivative consuming" in the sense that for a given order j_0, the coefficients $(\mathcal{B}_{j_0}^+, \mathcal{B}_{j_0}^-)$ are functions of the derivatives of order $j_0 - j$ of previous coefficients $(\mathcal{B}_j^+, \mathcal{B}_j^-)$. In particular, a theoretical accuracy of order δ^3 necessitates to compute accurately the second order derivatives of $\mathcal{B}_0^+$ as well as the curvatures of the interface Γ, and it is even worse for higher order, which leads to numerical instabilities. In addition, in most of the cases only the magnetic potential in the dielectric material is of great interest, while the expansion needs to compute also the potential in the conductor.

Therefore it may be crucial to find a numerical way to obtain the coefficients $(\mathcal{B}_j^+)$ without computing neither the curvature of the domain nor the second order derivatives of $\mathcal{B}_0^+$. As it is shown in the following, this can be obtained using the fact that the solution of (1) is complex-valued, since $\Im(\alpha) \neq 0$ and in the particular case of a real-valued source term and a slighty change in the above expansion.

2 An *a priori* insignificant but numerically useful remark

It is worth noting that even if the source J^+ located in the dielectric is a real-valued function, the coefficients $(\mathcal{B}_j^+, \mathcal{B}_j^-)$ are complex-valued, due to the transmission across the interface Γ. However, we show that if we change the expansion into

$$\mathcal{A}_\delta^+ \sim \sum_{l\geqslant 0} (\delta/\alpha)^l \, \mathcal{A}_l^+, \tag{4}$$

$$\mathcal{A}_\delta^- \sim \sum_{l\geqslant 0} (\delta/\alpha)^l \, \mathcal{A}_l^-(\mathbf{x};\delta) \quad \text{with} \quad \mathcal{A}_l^-(\mathbf{x};\delta) = \chi(\eta)\mathrm{v}_l(\mathbf{x_T}, \frac{\eta}{\delta}), \tag{5}$$

and if the source is real-valued, then all the coefficients $\mathcal{A}_j^+$ are real as shown in the next section. Such a result is intuitive since we easily check that $\mathcal{A}_0^+$ and $\mathcal{A}_1^+$ are given by

$$\left\{\begin{array}{lcll} -\Delta\mathcal{A}_0^+ &=& J^+ & \text{in } \Omega_+ , \\ \mathcal{A}_0^+ &=& 0 & \text{on } \Gamma, \\ \mathcal{A}_0^+ &=& 0 & \text{on } \partial\Omega , \end{array}\right. \qquad \left\{\begin{array}{lcll} -\Delta\mathcal{A}_1^+ &=& 0 & \text{in } \Omega_+ , \\ \mathcal{A}_1^+ &=& -\partial_{\mathbf{n}}\mathcal{A}_0^+ & \text{on } \Gamma, \\ \mathcal{A}_1^+ &=& 0 & \text{on } \partial\Omega . \end{array}\right. \tag{6}$$

Such a result is seemingly insignificant especially from the theoretical point of view since it is obvious that similar estimates as (2) hold with $\mathcal{A}_l^+$ instead of $\mathcal{B}_l^+$. However it becomes interesting for numerical purposes.

Actually, assume that $\mathcal{A}_0^+$ is computed, and let δ_0 satisfy

- δ_0 is small enough so that the estimate (2) holds for $k = 2$.
- δ_0 is not too small such that the solution of the whole problem (1) can be computed with an accuracy of order $O(\delta_0^3)$. We denote by $\mathcal{A}^+_{\delta_0,num}$ the corresponding numerical magnetic potential.

Thus for any $\delta \in (0, \delta_0)$, we can write the expansion of $\mathcal{A}^+_\delta$:

$$\mathcal{A}^+_\delta = \mathcal{A}^+_0 + \frac{\delta}{\alpha}\mathcal{A}^+_1 + \frac{\delta^2}{\alpha^2}\mathcal{A}^+_2 + O(\delta^3), \tag{7}$$

and we also have

$$\mathcal{A}^+_{\delta_0} = \mathcal{A}^+_{\delta_0,num} + O(\delta_0^3). \tag{8}$$

Then by identifying the above equalities for $|\delta/\alpha| = \delta_0$, since δ_0 has been taken small enough, and using the fact that $\mathcal{A}^+_1$ and $\mathcal{A}^+_2$ are real-valued functions, we infer, if $\Im(\alpha) \neq 0$ the following formulae:

$$\mathcal{A}^+_2 = -\frac{1}{\delta_0^2}\frac{|\alpha|^2}{\Im(\alpha)}\Im\left(\alpha\left(\mathcal{A}^+_{\delta_0,num} - \mathcal{A}^+_0\right)\right), \tag{9a}$$

$$\mathcal{A}^+_1 = \frac{1}{\delta_0}\Re\left(\alpha\left(\mathcal{A}^+_{\delta_0,num} - A^+_0\right)\right) - \delta_0\frac{\Re(\alpha)}{|\alpha|^2}\mathcal{A}^+_2, \tag{9b}$$

and then for any $\delta \in (0, \delta_0)$, the magnetic potential $\mathcal{A}^+_\delta$ is approached with an accuracy of $O(\delta_0^3)$:

$$\mathcal{A}^+_\delta = \mathcal{A}^+_0 + \frac{\delta}{\alpha}\mathcal{A}^+_1 + \frac{\delta^2}{\alpha^2}\mathcal{A}^+_2 + O(\delta_0^3). \tag{10}$$

The interesting point lies in the fact that we provided an approximation of $\mathcal{A}^+_\delta$, which is accurate at the order δ_0^3 without any computation of the curvature of the interface nor of the derivatives of the coefficients $\mathcal{A}^+_{0,1,2}$. Therefore with the 2 computations of $\mathcal{A}_{\delta_0,num}$ and $\mathcal{A}^+_0$, we extract the first three coefficients of the asymptotic expansion of $\mathcal{A}^+_\delta$, and thus we can make the parameters δ and α evolve in the range $|\delta/\alpha| < \delta_0$. If $\Im(\alpha) = 0$ then $\mathcal{A}^+_{\delta_0,num}$ is real and expressions (9) become

$$\mathcal{A}^+_2 = -\frac{|\alpha|^2}{\delta_0^2}\left(\mathcal{A}^+_{\delta_0,num} - \mathcal{A}^+_0\right), \tag{11a}$$

$$\mathcal{A}^+_1 = \frac{2\alpha}{\delta_0}\left(\mathcal{A}^+_{\delta_0,num} - \mathcal{A}^+_0\right). \tag{11b}$$

If in addition to $\mathcal{A}^+_0$ and $\mathcal{A}^+_{\delta_0,num}$, we also have $\mathcal{A}^+_1$ solution to (6), we can pushforward the reasoning to obtain the third coefficient $\mathcal{A}^+_3$:

$$\mathcal{A}^+_3 = -\frac{1}{\delta_0^3}\frac{|\alpha|^2}{\Im(\alpha)}\Im\left(\alpha^2\left(\mathcal{A}^+_{\delta_0,num} - \mathcal{A}^+_0 - \frac{\delta_0}{\alpha}\mathcal{A}^+_1\right)\right), \tag{12a}$$

$$\mathcal{A}^+_2 = \frac{1}{\delta_0^2}\Re\left(\alpha^2\left(\mathcal{A}^+_{\delta_0,num} - \mathcal{A}^+_0 - \frac{\delta_0}{\alpha}\mathcal{A}^+_1\right)\right) - \delta_0\frac{\Re(\alpha)}{|\alpha|^2}\mathcal{A}^+_3. \tag{12b}$$

Of course this is restricted to the condition that the interface Γ is smooth enough (at least of class C^3), to avoid any geometric singularity, which would interefere with the asymptotic expansions.

If α is real, then

$$\mathcal{A}_3^+ = -\frac{|\alpha|}{\delta_0^3}\left(2\alpha\left(\mathcal{A}^+_{\delta_0,num} - \mathcal{A}_0^+\right) - \delta_0\mathcal{A}_1^+\right), \tag{13a}$$

$$\mathcal{A}_2^+ = \frac{|\alpha|^2}{\delta_0^2}\left(\mathcal{A}^+_{\delta_0,num} - \mathcal{A}_0^+ - \frac{\delta_0}{\alpha}\mathcal{A}_1^+\right) - \delta_0\frac{\alpha}{|\alpha|^2}\mathcal{A}_3^+. \tag{13b}$$

In section 3, we show that the above coefficients $\mathcal{A}_l^+$ are indeed real-valued functions if J^+ is real-valued. Note that if J^+ is complex-valued, then the above formulae hold, writing the source terme as $J^+ = \Re(J^+) + \mathrm{j}\Im(J^+)$, with $\mathrm{j}^2 = -1$. Let illustrate the numerical efficiency of our remark.

2.1 Numerical efficiency of the method

Consider the geometric configuration of Fig. 1. Homogeneous zero Dirichlet condition is imposed on the purple part of the boundary of the conductor $\partial\Omega_c$, and on $\partial\Omega_d$ the imposed value of the magnetic potential is 1, elsewhere on $\partial\Omega$, homogeneous Neumann condition is imposed, and no volumic source term is imposed. In the following, δ equals $(f\sigma\mu)^{-1/2}$, where σ and μ are the respective conductivity and permeability of the conductor, both equal to 1, and $\alpha = \mathrm{j}$.

We define as f_{FE} the "low" frequency for which the computation of the whole problem is possible with a satisfactory accuracy, and we denote by $\delta_0 = 1/\sqrt{\sigma\mu f_{\mathrm{FE}}}$, and thus

$$\delta = \sqrt{f_{\mathrm{FE}}/f}\,\delta_0.$$

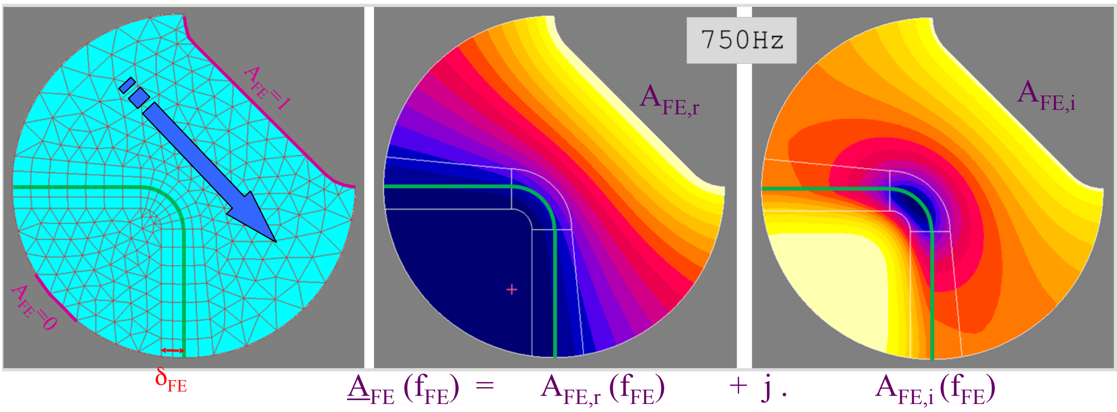

Figure 1: Geometric configuration and isovalues of the real and imaginary parts of reference solution obtained by a fine mesh.

Fig. 1 presents the quadratic error between the reference solution and 6 approximations:

- the coarse mesh, which made it possible to compute $\mathcal{A}^+_{\delta_0,num}$ at low frequency with a "satisfactory" accuracy, possibly of order δ_0^4
- the perfect conductor, which is coefficient $\mathcal{A}_0^+$
- the classical impedance boundary condition of Leontovitch
- the first-order approximation $\mathcal{A}_0^+ + \delta\mathcal{A}_1^+$

- the 2nd order appoximation obtained by expression (9)
- the 3rd order approximation obtained thanks to (12)

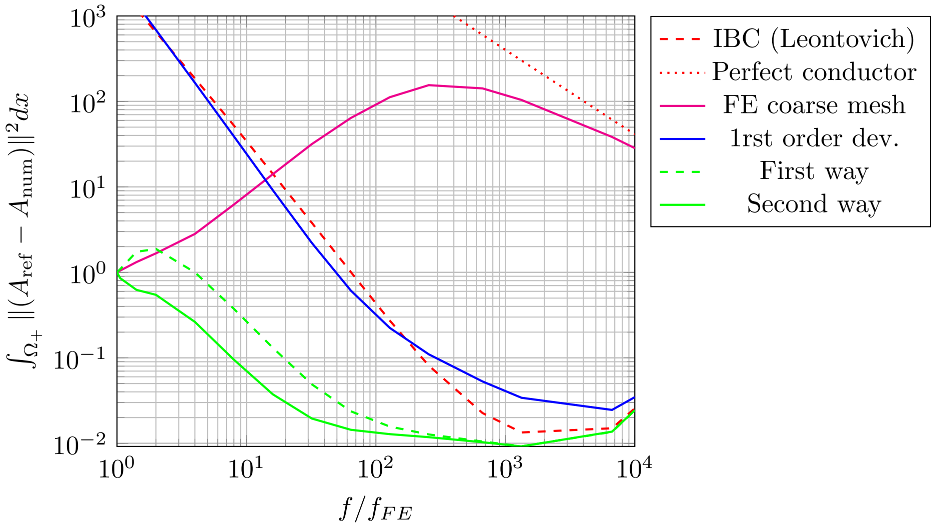

Figure 2: Quadratic error in the dielectric obtained by 6 approximations: the coarse mesh, the perfect conductor, the impedance boundary condition of Leontovitch, the first-order approximation $\mathcal{A}_0^+ + \delta\mathcal{A}_1^+$ and the 2nd and 3rd order approximation by identification (9) and (12) respectively.

As expected, the coarse mesh computation does not provide a good approximation of the potential. The perfect conductor approximation is also far from the reference solution, due to the fact that our frequencies are not high enough. First order approximation and the classical Leontovitch approximation give accurate estimates for frequencies 100 times higher that the reference frequency. Our two approximations are the most relevant, and provide a good approximation at any (almost) frequency higher than the reference frequency. It is worth noting that our numerical example is suboptimal compared to our expansion: this is probably due to the fact that our geometric configuration is not smooth enough. Indeed, the curvature of the interface jumps between 0 of the flat part to 1 along the circular case. However the numerical results are convincing and the approximation is accurate.

In the following section, we prove the assertion that all the coefficients $\mathcal{A}_j^+$ of (4) are real-valued functions.

3 Derivation of the real-valued coefficients of the asymptotics

Throughout the section, we assume that the source term J^+ is a real-valued function, smooth enough so that expansion (2) holds at least for $k \in \mathbb{N}$. We also focus on the case $d = 3$ but the case $d = 2$ is similar and even easier, *mutatis mutandis*.

We first remind the way the asymptotics are derived. The main idea is to observe that in the conductor, the magnetic potential decreases exponentially fast with respect to the normal variable to the interface Γ, and therefore the magnetic potential $\mathcal{A}_\delta^-$ is confined in a thin "boundary" layer $\mathcal{O}_\delta$ of thickness δ. Let us recall the geometrical framework useful for the derivation.

3.1 Geometry

Let $\mathbf{x_T} = (x_1, x_2)$ be a system of local coordinates on $\Gamma = \{\psi(\mathbf{x_T})\}$. Define the map Φ by

$$\forall(\mathbf{x_T}, x_3) \in \Gamma \times \mathbb{R}, \quad \Phi(\mathbf{x_T}, x_3) = \psi(\mathbf{x_T}) + x_3\mathbf{n}(\mathbf{x_T}),$$

where $\mathbf{n}$ is the normal vector of Γ directed towards the conductor. The thin layer $\mathcal{O}_\delta$ in which the magnetic potential is confined can then be parameterized by

$$\mathcal{O}_\delta = \{\Phi(\mathbf{x_T}, x_3), \quad (\mathbf{x_T}, x_3) \in \Gamma \times (0, \delta)\}.$$

The Euclidean metric in $(\mathbf{x_T}, x_3)$ is given by the 3×3–matrix $(g_{ij})_{i,j=1,2,3}$ where $g_{ij} = \langle \partial_i \Phi, \partial_j \Phi \rangle$:

$$\forall\alpha \in \{1,2\}, \quad g_{33} = 1,\ g_{\alpha 3} = g_{3\alpha} = 0, \tag{14a}$$

$$\forall(\alpha,\beta) \in \{1,2\}^2, \quad g_{\alpha\beta}(\mathbf{x_T}, x_3) = g^0_{\alpha\beta}(\mathbf{x_T}) + 2x_3 b_{\alpha\beta}(\mathbf{x_T}) + x_3^2 c_{\alpha\beta}(\mathbf{x_T}), \tag{14b}$$

where

$$g^0_{\alpha\beta} = \langle \partial_\alpha \psi, \partial_\beta \psi \rangle, \quad b_{\alpha\beta} = \langle \partial_\alpha n, \partial_\beta \psi \rangle, \quad c_{\alpha\beta} = \langle \partial_\alpha n, \partial_\beta n \rangle. \tag{14c}$$

We denote by (g^{ij}) the inverse matrix of (g_{ij}), and by g the determinant of (g_{ij}). For all $l \geq 0$ define

$$\begin{cases} \mathfrak{a}^l_{ij} = (-1)^l\ \partial_3^l \left(\dfrac{\partial_i\left(\sqrt{g} g^{ij}\right)}{\sqrt{g}} \right)\Bigg|_{x_3=0}, & \text{for } (i,j) \in \{1,2,3\}^2, \\ \mathfrak{b}^l_{\alpha\beta} = (-1)^l\ \partial_3^l \left(g^{\alpha\beta}\right)\big|_{x_3=0}, & \text{for } (\alpha,\beta) \in \{1,2\}^2, \end{cases} \tag{15}$$

and we denote by $\mathcal{S}^l_\Gamma$ the differential operator on Γ of order 2 defined by

$$\mathcal{S}^{-1}_\Gamma = 0, \qquad \mathcal{S}^l_\Gamma = \sum_{\alpha,\beta=1,2} \mathfrak{a}^l_{\alpha\beta}\partial_\beta + \mathfrak{b}^l_{\alpha\beta}\partial_\alpha\partial_\beta. \tag{16}$$

The key-point of the reasoning lies in the fact that in $\mathcal{O}_\delta$, we can use rescaled local coordinates $(\eta, \mathbf{x_T}) = (x_3/\delta, \mathbf{x_T})$ in order to write the Laplace operator as follows:

$$\begin{aligned} \Delta &= \frac{1}{\sqrt{g}} \sum_{i,j=1,2,3} \partial_i\left(\sqrt{g} g^{ij} \partial_j\right), \qquad \forall(\mathbf{x_T}, x_3) \in \Gamma \times (0,\delta) \\ &= \frac{1}{\delta^2}\partial_\eta^2 + \frac{1}{\delta}\mathfrak{a}^0_{33}(\mathbf{x_T})\partial_\eta + \sum_{l\geq 0} \delta^l \frac{\eta^l}{l!}\mathfrak{D}_l, \qquad \forall(\mathbf{x_T}, \eta) \in \Gamma \times (0,1), \end{aligned} \tag{17}$$

where $\mathfrak{D}_l$ are the first-order operator in η and second order in $\mathbf{x_T}$ given for $l \geq -1$ by

$$\mathfrak{D}_{-1} = \eta\mathfrak{a}^0_{33}(\mathbf{x_T})\partial_\eta, \qquad \forall l \geq 0, \quad \mathfrak{D}_l = \left(\frac{\eta}{l+1}\mathfrak{a}^{l+1}_{33}(\mathbf{x_T})\partial_\eta + \mathcal{S}^l_\Gamma \right). \tag{18}$$

We refer to [2] for the justification of such an expansion. In particular, it is worth noting that the function $\mathfrak{a}_{33}^l$ and $\mathfrak{b}_{\alpha\beta}^l$ comes from the geometry of the surface. For instance $\mathfrak{a}_{33}^0$ is the mean curvature of Γ. Then in the conductor, we denote by v_δ the solution to

$$\left\{\begin{array}{lcll} (-\partial_\eta^2+\alpha^2)\mathrm{v}_\delta - \delta^2 \displaystyle\sum_{n\geqslant -1} \delta^n \frac{\eta^n}{n!}\mathfrak{D}_n(\mathrm{v}_\delta) & = & 0 & \text{in} \quad \Gamma\times(0,+\infty), \\ \partial_\eta \mathrm{v}_\delta|_{\eta=0} & = & \delta\partial_{\mathbf{n}}\mathcal{A}_\delta^+|_\Gamma & \text{on} \quad \Gamma\times\{0\}\ . \end{array}\right. . \tag{19}$$

We insert the Ansatz

$$\mathcal{A}_\delta^+ \sim \sum_{k\geqslant 0} (\delta/\alpha)^k \mathcal{A}_k^+ \quad \text{and} \quad v_\delta \sim \sum_{k\geqslant 0}(\delta/\alpha)^k \mathrm{v}_k(\mathbf{x_T},\eta/\delta),$$

into (19) and we perform the identification of terms with the same power in δ/α. The term $\mathcal{A}_k^+$, and v_k satisfy:

$$(-\partial_\eta^2+\alpha^2)\mathrm{v}_k = \sum_{l=-1}^{k-2} \alpha^{l+2}\frac{\eta^l}{l!}\mathfrak{D}_l(\mathrm{v}_{k-2-l}) \qquad \text{for} \quad 0<\eta<+\infty\ , \tag{20a}$$

$$\partial_\eta \mathrm{v}_k = \alpha\partial_{\mathbf{n}}\mathcal{A}_{k-1}^+ \qquad \text{for} \quad \eta=0\ , \tag{20b}$$

$$\lim_{\eta\to+\infty} \mathrm{v}_k = 0\,, \tag{20c}$$

and

$$-\Delta\mathcal{A}_k^+ = \delta_{k,0}J^+, \qquad \text{in} \quad \Omega_+\ , \tag{20d}$$

$$\mathcal{A}_k^+ = \mathrm{v}_k|_{\eta=0}, \qquad \text{on} \quad \Gamma\ , \tag{20e}$$

$$\mathcal{A}_\delta^+ = 0, \qquad \text{on} \quad \partial\Omega\ , \tag{20f}$$

Simple calculations show that $\mathrm{v}_0\equiv 0$ and $\mathcal{A}_0^+$ is real-valued since it satisfies

$$\left\{\begin{array}{lcll} -\Delta\mathcal{A}_0^+ & = & J^+ & \text{in} \quad \Omega_+\ , \\ \mathcal{A}_0^+ & = & 0 & \text{on} \quad \Gamma\ , \\ \mathcal{A}_0^+ & = & 0 & \text{on} \quad \partial\Omega\ . \end{array}\right. \tag{21}$$

One step further, one has $\mathrm{v}_1(\mathbf{x_T},\eta) = -e^{-\alpha\eta}\partial_{\mathbf{n}}\mathcal{A}_0^+|_\Gamma$ and $\mathcal{A}_1^+$, defined by

$$\left\{\begin{array}{lcll} -\Delta\mathcal{A}_1^+ & = & 0 & \text{in} \quad \Omega_+\ , \\ \mathcal{A}_1^+ & = & -\partial_{\mathbf{n}}\mathcal{A}_0^+|_\Gamma & \text{on} \quad \Gamma\ , \\ \mathcal{A}_1^+ & = & 0 & \text{on} \quad \partial\Omega\ , \end{array}\right. \tag{22}$$

is a real-valued function.

By induction, we prove the following proposition:

Proposition 3.1. *For any $k\geq 0$ the functions $(\mathrm{v_k},\mathcal{A}_k^+)$ defined by (20) satisfy*

$$(H_k) \qquad : \qquad \begin{cases} \mathcal{A}_k^+ \text{ is a real-valued function in } \Omega_+, \\ \mathrm{v}_k(\mathbf{x_T},\eta) = e^{-\alpha\eta}\sum_{l=0}^{k-1} a_{k,l}(\mathbf{x_T})(\alpha\eta)^l, \\ \text{where } a_{k,l} \text{ are real-valued functions of the tangential variable } \mathbf{x_T}. \end{cases}$$

Proof. As mentioned above, (H_0) and (H_1) are true. Suppose that (H_l) is true up to the rank $k-1 \geq 0$, let show that (H_k) holds. Since $\mathrm{v}_0 \equiv 0$, v_k satisfies the second order ordinary differential equation in η:

$$(-\partial_\eta^2 + \alpha^2)\mathrm{v}_k = \sum_{l=-1}^{k-3} \alpha^{l+2} \frac{\eta^l}{l!} \mathfrak{D}_l(\mathrm{v}_{k-2-l}) \qquad \text{for} \quad 0 < \eta < +\infty \ , \tag{23}$$

$$\partial_\eta \mathrm{v}_k = \alpha \partial_{\mathbf{n}} \mathcal{A}_{k-1}^+ \qquad \text{for} \quad \eta = 0 \ , \tag{24}$$

$$\lim_{\eta \to +\infty} \mathrm{v}_k = 0 \,, \tag{25}$$

with the convention $l! = 1$ for $\in \{-1, 0, 1\}$. We just have to exhibit the solution, which is necessary unique. We thus look for a solution as

$$\mathrm{v}_k = e^{-\alpha\eta} \sum_{l=0}^{k-1} a_{k,l}(\mathbf{x_T})(\alpha\eta)^l,$$

and we aim at determining $(a_{k,l})_{l=0}^{k-1}$ in terms of the coefficients $(a_{k-l,n})_{n=0}^{k-l-1}$, for $l = 1, k-1$, which are assumed to be known by hypothesis. The difficulty lies in the fact that the right-hand side term is quite tricky to address. However, using the explicit expression of $\mathfrak{D}_l$ and the hypothesis on the form of the functions v_l for $l = 0, \cdots, k-1$ one infers:

$$\begin{aligned} e^{\alpha\eta}\mathfrak{D}_l(\mathrm{v}_{k-2-l}) = {} & -(\alpha\eta)^{k-l-2} \frac{\mathfrak{a}_{33}^{l+1}}{l+1} a_{k-l-2,k-l-3} \\ & + \sum_{n=1}^{k-l-3} (\alpha\eta)^n \left\{ \frac{\mathfrak{a}_{33}^{l+1}}{l+1} (n a_{k-l-2,n} - a_{k-l-2,n-1}) + \mathcal{S}_\Gamma^l(a_{k-l-2,n}) \right\} \\ & + \mathcal{S}_\Gamma^l(a_{k-l-2,0}). \end{aligned}$$

Therefore, the coefficients of the right-hand side term of (23) can be ordered as follows

$$\begin{aligned} \sum_{l=-1}^{k-3} \alpha^{l+2} \frac{\eta^l}{l!} \mathfrak{D}_l(\mathrm{v}_{k-2-l}) = \alpha^2 e^{-\alpha\eta} \Bigg[& -(\alpha\eta)^{k-2} \sum_{l=-1}^{k-3} \frac{\mathfrak{a}_{33}^{l+1}}{(l+1)!} a_{k-l-2,k-l-3} \\ & + \sum_{q=0}^{k-3} (\alpha\eta)^q \Bigg\{ \sum_{p=-1}^{q-1} \frac{1}{p!} \Bigg[\frac{\mathfrak{a}_{33}^{p+1}}{p+1} \big((q-p) a_{k+p-2,q-p} \\ & \qquad - a_{k+p-2,q-p-1} \big) \\ & \qquad + \mathcal{S}_\Gamma^p(a_{k-p-2,q-p}) \Bigg] + \frac{1}{q!} \mathcal{S}_\Gamma^q(a_{k-q-2,0}) \Bigg\} \Bigg] \end{aligned} \tag{26}$$

On the other hand, simple calculations make us see that

$$\begin{aligned} -\partial_\eta^2 \mathrm{v}_k + \alpha^2 \mathrm{v}_k = \alpha^2 e^{-\alpha\eta} \Bigg\{ & 2(k-1)(\alpha\eta)^{k-2} a_{k,k-1} \\ & - \sum_{q=0}^{k-3} (q+1) \Big((q+2) a_{k,q+2} - 2 a_{k,q+1} \Big) (\alpha\eta)^q \Bigg\} \end{aligned} \tag{27}$$

Now it just remains to identify the terms with the power of $(\alpha\eta)$ in (26) and (27) to infer the following inductive relations:

$$2(k-1)a_{k,k-1} = -\sum_{l=-1}^{k-3} \frac{\mathfrak{a}_{33}^{l+1}}{(l+1)!} a_{k-l-2,k-l-3}, \tag{28a}$$

for any $0 \le q \le k-3$

$$\begin{aligned} -(q+1)\left((q+2)a_{k,q+2} - 2a_{k,q+1}\right) = \sum_{p=-1}^{q-1} \frac{1}{p!} \Bigg[\frac{\mathfrak{a}_{33}^{p+1}}{p+1} \big((q-p)a_{k+p-2,q-p} \\ - a_{k+p-2,q-p-1}\big) \\ + \mathcal{S}_\Gamma^p (a_{k-p-2,q-p}) \Bigg] + \frac{1}{q!} \mathcal{S}_\Gamma^q (a_{k-q-2,0}). \end{aligned} \tag{28b}$$

We eventually use the condition (24) and the fact that

$$\partial_\eta \mathrm{v}_k|_{\eta=0} = \alpha \left(a_{k,1} - a_{k,0}\right),$$

to infer the last condition:

$$a_{k,1} - a_{k,0} = \partial_n \mathcal{A}_{k-1}^+|_{\Gamma^+}. \tag{28c}$$

By hypothesis, all the right-hand side of (28) are real, and the system satisfied by $(a_{k,l})_{l=0}^{k-1}$ is clearly invertible so (H_k) holds and the proposition is shown.

□

References

[1] H. Haddar, P. Joly, and H.-M. Nguyen. Generalized impedance boundary conditions for scattering by strongly absorbing obstacles: The scalar case. *Mathematical Models and Methods in Applied Sciences*, 15(08):1273–1300, 2005.

[2] R. Perrussel and C. Poignard. Asymptotic expansion of steady-state potential in a high contrast medium with a thin resistive layer. *Applied Mathematics and Computation*, 221(0):48 – 65, 2013.

[3] S.M. Rytov. Calculation of skin effect by perturbation method. *Journal Experimenal'noi i Teoreticheskoi Fiziki*, 10(2), 1940.

[4] T.B.A. Senior and J.L. Volakis. Derivation and application of a class of generalized impedance boundary conditions. *IEEE Trans. on Antennas and Propagation*, 37(12), 1989.

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